HARDY-TYPE INEQUALITIES FOR LINEAR DIFFERENTIAL OPERATOR AND WIDDER'S DERIVATIVES

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ABSTRACT*: In this paper, we give a wide range of Hardy-type inequalities for linear differential operator and Widder's derivatives. As an application of Hardy-type inequalities we extract the inequalities related to inequality of G. H. Hardy involving linear differential operator and Widder's derivatives.*

Keywords: Inequalities; kernel; Green function; linear differential operator; Widder's derivative.

INTRODUCTION

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures. Let $U(f,k)$ denote the class of function $g : \Omega_1 \to \mathbb{R}$ belongs to the class if it admits the representation

$$g(x) = \int_{\Omega_2} k(x,t) f(t) d\mu_2(t)$$

and A_k be an integral operator defined by

$$A_k f(x) := \frac{g(x)}{K(x)} = \frac{1}{K(x)} \int_{\Omega_2} k(x,t) f(t) d\mu_2(t), \tag{1.1}$$

where $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable and non-negative kernel, $f : \Omega_2 \to R$ is measurable function on Ω_2 and

$$0 < K(x) := \int_{\Omega_2} k(x,t) d\mu_2(t), \quad x \in \Omega_1, \tag{1.2}$$

Čizmešija, Krulić, Pečarić and Persson has established a lot of Hardy-type inequalities in their recent papers [6], [7] which is a remarkable contribution in theory of inequalities. But our purpose is to give such Hardy-type inequalities for linear differential operator and Widder's derivative.

The following theorem is give in [6](see also [2]).

Theorem 1.1. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_2 by (1.2). Suppose that the function $x \mapsto u(x)\dfrac{k(x,t)}{K(x)}$ is integrable on Ω_1 for each fixed $t \in \Omega_2$ and v is defined on Ω_1 by

$$v(t) := \int_{\Omega_1} \frac{u(x)k(x,t)}{K(x)} d\mu_1(x) < \infty. \tag{1.3}$$

If Φ is convex on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_{\Omega_1} u(x)\Phi\left(A_k f(x)\right) d\mu_1(x) \le \int_{\Omega_2} v(t)\Phi\left(f(t)\right) d\mu_2(t),$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$, such that $Imf \subseteq I$, where A_k is defined by (1.1).

Substitute $k(x,t)$ by $k(x,t) f_2(t)$ and f by $\dfrac{f_1}{f_2}$, where $f : \Omega_i \to \mathbb{R}$, $(i = 1,2)$ are measurable functions in Theorem 1.1 we obtain the following result (see [4]).

Theorem 1.2. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$. Assusme that the function $x \mapsto u(x)\dfrac{k(x,t)}{g_2(x)}$ is integrable on Ω_1 for each fixed $t \in \Omega_2$. Defined v on Ω_2 by

$$v(t) := f_2(t) \int_{\Omega_1} \frac{u(x)k(x,t)}{g_2(x)} d\mu_1(x) < \infty. \tag{1.4}$$

If $\Phi : I \to \mathbb{R}$ is a convex function and $\dfrac{g_1(x)}{g_2(x)}, \dfrac{f_1(x)}{f_2(x)} \in I$, then the inequality

$$\int_{\Omega_1} u(x)\Phi\left(\frac{g_1(x)}{g_2(x)}\right) d\mu_1(x) \le \int_{\Omega_2} v(t)\Phi\left(\frac{f_1(x)}{f_2(x)}\right) d\mu_2(t),$$

holds for all $g_i \in U(f_i,k)$, $(i = 1,2)$ and for measurable functions $f : \Omega_i \to \mathbb{R}$, $(i = 1,2)$.

The following theorem is given in [3].

Theorem 1.3. Let u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_2 by (1.2). Assume that the function $x \mapsto u(x)\dfrac{k(x,t)}{K(x)}$ is integrable on Ω_1 for each fixed $t \in \Omega_2$ and that v is defined on Ω_2 by (1.3). If $\Phi:(0,\infty) \to R$ is a convex and increasing function, then the inequality

$$\int_{\Omega_1} u(x)\Phi\left(\frac{g(x)}{K(x)}\right) d\mu_1(x) \le \int_{\Omega_2} v(t)\Phi\left(f(t)\right) d\mu_2(t),$$

holds for all measurable functions $f:\Omega_2 \to \mathrm{R}$, and for all $g \in U(f,k)$.

Substitute $k(x,t)$ by $k(x,t)f_2(t)$ and f by $\dfrac{f_1}{f_2}$, where $f:\Omega_i \to \mathrm{R}$, $(i=1,2)$ are measurable functions in Theorem 1.3, we obtain the following result (see [5]).

Theorem 1.4. Let $f_i:\Omega \to R$ be measurable functions, $g_i \in U(f_i,k)$, $(i=1,2)$, where $g_2(x)>0$ for every $x \in \Omega_1$. Let u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$. Assusme that the function $x \mapsto u(x)\dfrac{f_2(t)k(x,t)}{K(x)}$ is integrable on Ω_1 for each fixed $t \in \Omega_2$. Defined v on Ω_2 by (1.4). If $\Phi:(0,\infty) \to \mathrm{R}$ is a convex and incresing function, then the inequality

$$\int_{\Omega_1} u(x)\Phi\left(\left|\frac{g_1(x)}{g_2(x)}\right|\right) d\mu_1(x) \le \int_{\Omega_2} v(t)\Phi\left(\left|\frac{f_1(x)}{f_2(x)}\right|\right) d\mu_2(t),$$

holds.

Remark 1.5. If Φ is strictly convex and $\frac{f_1(x)}{f_2(x)}$ is non-constant, then inequalities given in Theorem 1.2 and Theorem 1.4 are strict.

Theorem 1.6. Let $0<p\le q<\infty$ and let the assumptions of Theorem 1.1 be satisfied but now with

$$v(t) := \left(\int_{\Omega_1} u(x)\left(\frac{k(x,t)}{K(x)}\right)^{\frac{q}{p}} d\mu_1(x)\right)^{\frac{p}{q}} < \infty$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathrm{R}$, then the inequality

$$\left(\int_{\Omega_1} u(x)\left[\Phi\left(A_k f(x)\right)\right]^{\frac{q}{p}} d\mu_1(x)\right)^{\frac{1}{q}} \le \left(\int_{\Omega_2} v(y)\Phi\left(f(y)\right) d\mu_2(y)\right)^{\frac{1}{p}},$$

holds for all measurable functions $f:\Omega_2 \to \mathrm{R}$ such that $\operatorname{Im} f \subseteq I$, where A_k is defined by (1.1).

The upcoming theorem is given in [7].

Theorem 1.7. Let $0<p\le q<\infty$. Let $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1, ω be a μ_2-a.e. positiv function on Ω_2, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_1 by (1.2). Suppose that the $K(x)>0$ for all $x \in \Omega_1$ and that the function $x \mapsto u(x)\left(\dfrac{k(x,t)}{K(x)}\right)^{\frac{q}{p}}$ is integrable on Ω_1 for each fixed $t \in \Omega_2$. Let Φ be a non-negative convex function on an interval $I \subseteq \mathrm{R}$. If

$$A = \sup_{t \in \Omega_2} \omega^{\frac{-1}{p}}(t)\left(\int_{\Omega_1} u(x)\left(\frac{k(x,t)}{K(x)}\right)^{\frac{q}{p}} d\mu_1(x)\right)^{\frac{p}{q}} < \infty,$$ then

there exists a poistive real number C such that the inequality

$$\left(\int_{\Omega_1} u(x)\left[\Phi\left(A_k f(x)\right)\right]^{\frac{q}{p}} d\mu_1(x)\right)^{\frac{1}{q}} \le \left(\int_{\Omega_2} \omega(y)\Phi\left(f(y)\right) d\mu_2(y)\right)^{\frac{1}{p}}, \tag{1.5}$$

Holds for all measurable functions $f:\Omega_2 \to \mathrm{R}$ with the values in I and $A_k f$ be defined by (1.1). Moreover if C is smaller constant for (1.5) to hold, then $C \le A$.

Our analysis continues by providing a new two-parametric class of sufficient conditions for a weighted modular inequality involving the operator A_k to hold. The conditions obtained depend on a real parameter s and a positive function V on Ω_2.

Next result is given in [7].

Theorem 1.8. Let $1<p\le q<\infty$. Let $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1, v be a measurable $\mu_2 - a.e.$

positive function on Ω_2, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_1 by (1.2). Let $K(x) > 0$ for all $x \in \Omega_1$ and let the function

$x \mapsto u(x)\left(\frac{k(x,t)}{K(x)}\right)^{\frac{q}{p}}$ is integrable on Ω_1 for each fixed $t \in \Omega_2$. Suppose that $\Phi : I \to [0,\infty)$ is a bijective convex function on an interval $I \subseteq \mathrm{R}$. If there exists a real parameter $s \in (1, p)$ and positve measurable function $V : \Omega_2 \to \mathrm{R}$ such that

$$A(s,V) = F(V,v)\sup_{t\in\Omega_2} V^{\frac{s-1}{p}}(y)$$

$$\times\left(\int_{\Omega_1} u(x)\left(\frac{k(x,t)}{K(x)}\right)^q d\mu_1(x)\right)^{\frac{1}{q}} < \infty$$

$$\|f\|_v(a,b) \le \left(\frac{K_5(a)}{(g_n(b))^{1-v}}\right)^{\frac{1}{v}} \|L_{n+1} f\|_v(a,b).$$

where

$$F(V,v) = \left(V^{\frac{-p'(s-1)}{p}}(t)v^{1-p'}(t)d\mu_2(t)\right)^{\frac{1}{p'}}$$

and p' is exponent conjugate of p, thre exists a positive real constant C such that the inequality

$$\left(\int_{\Omega_1} u(x)\left[\Phi\left(A_k f(x)\right)\right]^q d\mu_1(x)\right)^{\frac{1}{q}}$$

$$\le C\left(\int_{\Omega_2} \omega(y)\Phi^p\left(f(y)\right)d\mu_2(y)\right)^{\frac{1}{p}}, \qquad (1.6)$$

holds for all measurable function $f : \Omega_2 \to \mathrm{R}$ with the values in I and $A_k f$ be defined by (1.1). Moreover if C is smaller constant for (1.6) to hold, then

$$C \le \inf_{\substack{1<s<p \\ V>0}} A(s,V).$$

Modification of tTheorem 1.8 give the next result and is given in [7].

Theorem 1.9. Let $1 < p \le q < \infty$, $1 < s < p$, and $0 < b < \infty$. Let u be a weight function on $(0,b)$, ω be an a.e. positive measurable function on $(0,b)$, k be a non-negative measurable function on $(0,b)\times(0,b)$ and

$$0 < K(x) := \int_0^x k(x,t)dt, \quad x \in (0,b).$$

Let I be an interval in R and $\Phi : I \to [0,\infty)$ be a bijective convex function. If

$$V(t) = \int_0^t \omega^{1-p'}(x)x^{p'-1}dx < \infty$$

holds almost everywhere in $(0,b)$, and

$$A = \sup_{0<t<b}\left(\int_t^b u(x)\left(\frac{k(x,t)}{K(x)}\right)^q V^{\frac{q(p-s)}{p}}(x)\frac{dx}{x}\right)^{\frac{1}{q}}$$

$$\times V^{\frac{s-1}{p}}(t) < \infty,$$

then there exists a positive real constant C such that

$$\left(\int_0^b u(x)\left[\Phi\left(A_k f(x)\right)\right]^q \frac{dx}{x}\right)^{\frac{1}{q}}$$

$$\le C\left(\int_0^b \omega(x)\Phi^p\left(f(x)\right)\frac{dx}{x}\right)^{\frac{1}{p}}, \qquad (1.7)$$

holds for all measurable functions $f : (0,b) \to \mathrm{R}$ with the values in I and the Hardy-type operator $A_k f$ defined by

$$A_k f(x) := \frac{1}{K(x)}\int_0^x k(x,t)f(t)dt, \quad x \in (0,b). \qquad (1.8)$$

Moreover if C is the best possible constant in (1.7), then

$$C \le \inf_{1<s<p}\left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}} A(s).$$

New general refined weighted Hardy-type inequality with a non-negative kernel, related to an arbitrary non-negative convex function is given in the upcoming theorem and is given in [6].

Theorem 1.10 Let $\tau \in \mathrm{R}_+$, $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_2, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_1 by (1.2). Suppose that the $K(x) > 0$ for all $x \in \Omega_1$ and that thte function

$$x \mapsto u(x)\left(\frac{k(x,t)}{K(x)}\right)^{\tau}$$

is integrable on Ω_1 for each fixed $t \in \Omega_2$ and that v is defined by

$$v(t) := \left(\int_{\Omega_1} u(x)\left(\frac{k(x,t)}{K(x)}\right)^{\tau} d\mu_1(x)\right)^{\tau} < \infty$$

If Φ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int\, I$, then the inequality

$$\left(\int_{\Omega_2} v(t)\Phi\left(f(t)\right)d\mu_2(t)\right)^{\tau} - \left(\int_{\Omega_1} u(x)\left[\Phi\left(A_k f(x)\right)\right]^{\tau} d\mu_1(x)\right)$$
$$\geq \tau \int_{\Omega_1} \frac{u(x)}{K(x)}\Phi^{\tau-1}(A_k f(x)) \times \int_{\Omega_2} k(x,t)\, r(x,t) d\mu_2(t) d\mu_1(x) \quad (1.9)$$

holds for $\tau \geq 1$ and all measurable functions $f : \Omega_2 \to \mathbb{R}$ with the values in I, where $A_k f$ be defined by (1.1) and the function $r : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is defined by

$$r(x,t) = \|\left[\Phi(f(t)) - \Phi(A_k f(x))\right] - |\varphi(A_k f(x))| \cdot (f(t) - A_k f(x))\| \quad (1.10)$$

If $\tau \in (0,1]$ and the function $\Phi : I \to \mathbb{R}$ is positive and concave, then the order of term on the left hand side of inequality (1.9), is reversed, that is, the inequality

$$\left(\int_{\Omega_1} u(x)\left[\Phi\left(A_k f(x)\right)\right]^{\tau} d\mu_1(x)\right) - \left(\int_{\Omega_2} v(t)\Phi\left(f(t)\right)d\mu_2(t)\right)^{\tau} \geq \tau \int_{\Omega_1} \frac{u(x)}{K(x)}\Phi^{\tau-1}(A_k f(x))$$
$$\times \int_{\Omega_2} k(x,t)\, r(x,t) d\mu_2(t) d\mu_1(x)$$

holds.

Let the function $r_1 : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be defined by

$$r_1(x,t) = [\Phi(f(t)) - \Phi(A_k f(x)) - |\varphi(A_k f(x))| \cdot (f(t) - A_k f(x))] \quad (1.11)$$

If Φ is a non-negative monotone convex function on the interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in IntI$, then the inequality

$$\left(\int_{\Omega_2} v(t)\Phi\left(f(t)\right)d\mu_2(t)\right)^{\tau} - \left(\int_{\Omega_1} u(x)\left[\Phi\left(A_k f(x)\right)\right]^{\tau} d\mu_1(x)\right)$$
$$\geq \tau \int_{\Omega_1} \frac{u(x)}{K(x)}\Phi^{\tau-1}(A_k f(x)) \int_{\Omega_2} \operatorname{sgn}(f(t) - A_k f(x))$$
$$\times k(x,t)\, r(x,t) d\mu_2(t) d\mu_1(x) \quad (1.12)$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ such that $f(t) \in I$ for all fixed $t \in \Omega_2$ where $A_k f$ be defined by (1.1)

If Φ is non-negative monotone concave, then the order of the terms on the left hand side of (1.12) is reversed.

In [6], the next theorem with one dimensional is given.

Theorem 1.11. Let $0 < b \leq \infty$ and $k : (0,b) \times (0,b) \to \mathbb{R}$, $u : (0,b) \to \mathbb{R}$ be non-negative measurable function and

$$\omega(t) := t\left(\int_t^b u(x)\left(\frac{k(x,t)}{K(x)}\right)^{\frac{q}{p}} \frac{dx}{x}\right)^{\frac{p}{q}} < \infty, \quad t \in (0,b)$$

If $0 < p \leq q < \infty$, or $-\infty < q \leq p < 0$, Φ is non-negative convex function on the interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function $\varphi(x) \in \partial\Phi(x)$ for all $x \in IntI$, then the inequality

$$\left(\int_0^b \omega(t)\Phi\left(f(t)\right)\frac{dt}{t}\right)^{\frac{q}{p}} - \left(\int_0^b u(x)\left[\Phi\left(A_k f(x)\right)\right]^{\frac{q}{p}} \frac{dx}{x}\right)$$
$$\geq \frac{q}{p}\int_0^b \frac{u(x)}{K(x)}\Phi^{\frac{q}{p}-1}(A_k f(x)) \int_0^x k(x,t)\, r(x,t) dt \frac{dx}{x} \quad (1.13)$$

holds for all measurable functions $f : (0,b) \to \mathbb{R}$ with values in I, where $A_k f$ and r are respectively defined by (1.8) and (1.10).

If Φ is non-negative monotone convex function on the interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is that $\varphi(x) \in \partial\Phi(x)$ for all $x \in IntI$, then the inequality

$$\left(\int_0^b \omega(t)\Phi\left(f(t)\right)\frac{dt}{t}\right)^{\frac{q}{p}} - \left(\int_0^b u(x)\left[\Phi\left(A_k f(x)\right)\right]^{\frac{q}{p}} \frac{dx}{x}\right)$$
$$\geq \frac{q}{p}\left|\int_0^b \frac{u(x)}{K(x)}\Phi^{\frac{q}{p}-1}(A_k f(x))\right.$$
$$\left.\int_0^x \operatorname{sgn}(f(t) - A_k f(x)) k(x,t)\, r_1(x,t) dt \frac{dx}{x}\right| \quad (1.14)$$

holds for all measeurable functions $f : (0,b) \to \mathbb{R}$ such that $f(t) \in I$ for all fixed $t \in \Omega_2$ where $A_k f$ and r_1 be defined by (1.8) and (1.11) respectively .

If $0 < p \leq q < \infty$, or $-\infty < q \leq p < 0$, Φ is non-negative (monotone) convex function then (1.13) and (1.14) holds with reverse order of integrals on its left hand side.

The rest of the paper is organized in the following way: In Section 2, we prove new Hardy-type and refined Hardy-type inequalities involving linear differential operator. We also establish the inequalities related to inequality of G. H. Hardy. Section 3 deals with the Hardy-type, refined Hardy-type and inequality related to inequality of G. H. Hardy for Widder's derivative.

2.0 POINCARÉ LIKE INEQUALITIES FOR LINEAR DIFFERENTIAL OPERATOR

Let $[a,b] \subset \mathrm{R}$, $a_i(x), i = 1,\ldots,n-1 (n \in \mathrm{N})$, and $h(x)$ be continuous functions on $[a,b]$, and let

$$L = D^n + a_{n-1}(x)D^{n-1} + \ldots + a_0(x),$$

be a fixed linear differential operator on $C^n[a,b]$. Let $y_1(x),\ldots,y_n(x)$ be a set of linearly independent solution to $Ly = 0$ and the associated Green's function for L is

$$H(x,t) := \frac{\begin{vmatrix} y_1(t) & \cdot & \cdot & \cdot & y_n(t) \\ y_1'(t) & \cdot & \cdot & \cdot & y_n'(t) \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ y_1^{(n-2)}(t) & & & \cdot & y_n^{(n-2)}(t) \\ y_1(x) & \cdot & \cdot & \cdot & y_n(x) \end{vmatrix}}{\begin{vmatrix} y_1(t) & \cdot & \cdot & \cdot & y_n(t) \\ y_1'(t) & \cdot & \cdot & \cdot & y_n'(t) \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ y_1^{(n-2)}(t) & & & \cdot & y_n^{(n-2)}(t) \\ y_1(t) & \cdot & \cdot & \cdot & y_n(t) \end{vmatrix}} \qquad (2.1)$$

which is continuous function on $[a,b]^2$. Consider fixed a, then

$$y(x) = \int_a^b H(x,t)h(t)dt, \quad \text{for all} \quad x \in [a,b]$$

is the unique solution to the initial value problem

$$Ly = h, \quad y^{(i)}(a) = 0, \quad i = 0,1,\ldots,n-1.$$

Now we present the Poincaré like inequality for linear differential operators.

Theorem 2.1 Let u be a weight function on (a,b), H be a positive Green function associated to the linear differential operator L. Suppose that the function $x \mapsto u(x)\dfrac{H(x,t)}{\tilde{H}(x)}$ is integrable on (a,b) for each fixed t ∈ (a, b), and that v is defined on (a,b) by

$$v(t) := \int_{\Omega_1} \frac{u(x)H(x,t)}{\tilde{H}(x)} d\mu_1(x) < \infty. \qquad (2.2)$$

If Φ is convex on the interval $I \subseteq \mathrm{R}$, then the inequality

$$\int_a^b u(x)\Phi\left(\frac{1}{\tilde{H}(\mathrm{x})}\int_a^x H(\mathrm{x},\mathrm{t})h(\mathrm{t})dt\right)dx$$

$$\leq \int_a^b v(t)\Phi\big(h(t)\big)dt, \qquad (2.3)$$

holds for all measurable functions $f : (\mathrm{a},\mathrm{b}) \to \mathrm{R}$ such that $\operatorname{Im} h \subseteq I$, where $\tilde{H}$ is defined as:

$$\tilde{H}(\mathrm{x}) = \int_a^x H(\mathrm{x},\mathrm{t})\,\mathrm{dt} \qquad (2.4)$$

Proof. Applying Theorem 1.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$ and $k(x,t) = H(x,t)$ we obtain inequality (2.3).

We continue with the inequality of G. H. Hardy. Iqbal et.al. in their paper [3] proved the inequality of G. H. Hardy but here our purpose is to establish the inequality related to inequality of G. H. Hardy for linear differential operator.

Remark 2.2. Choose the particular convex function $\Phi(\mathrm{x}) = \mathrm{x}^{\nu}$, $\nu \geq 1$ and weight function $u(x) = \tilde{H}(x,t)$ in Theorem 2.1 we get

$$\int_a^b \tilde{H}(x)^{1-\nu}\left(\int_a^x H(x,t)h(t)dt\right)^{\nu} dx$$

$$\leq \int_a^b K_1(t)h^{\nu}(t)dt. \qquad (2.5)$$

Inequality (2.5) gives

$$\tilde{H}(b)^{1-\nu}\int_a^b y^{\nu}(x)dx \leq K_1(a)\int_a^b h^{\nu}(t)dt.$$

This implies that

$$\|y\|_{\nu}(a,b) \leq \left(\frac{K_1(a)}{(\tilde{H}(b))^{1-\nu}}\right)^{\frac{1}{\nu}} \|h\|_{\nu}(a,b).$$

If we substitute $H(x,t)$ by $H(x,t)h_2(t)$ and h by $\dfrac{h_1}{h_2}$, where $h_i : (a,b) \to \mathrm{R}$, $(i = 1,2)$ are measurable functions in Theorem 2.1 we obtain the following result.

Theorem 2.3 Let u be a weight function on (a,b), H be a positive Green function associated to the linear differential operator L. Suppose that the function $x \mapsto u(x)\dfrac{H(x,t)}{y_2(x)}$ is integrable on (a,b) for each fixed $t \in (a,b)$, and that w is defined on (a,b) by

$$w(t) := h_2(t)\int_t^b u(x)\frac{H(x,t)}{y_2(x)}dx < \infty. \qquad (2.6)$$

If $\Phi : I \to \mathrm{R}$ is a convex function and $\dfrac{y_1(x)}{y_2(x)}, \dfrac{h_1(t)}{h_2(t)} \in I$, then the inequality

$$\int_a^b u(x)\Phi\left(\frac{y_1(x)}{y_2(x)}\right)dx \le \int_a^b w(t)\Phi\left(\frac{h_1(t)}{h_2(t)}\right)dt, \qquad (2.7)$$

holds for all $y_i \in U(h_i, H), (i = 1,2)$ and for all measurable functions $h_i : (a,b) \to \mathrm{R}$, $(i = 1,2)$.

Here we give Hardy-type inequalities for linear differential operators involving convex and increasing function.

Theorem 2.4 Let u be a weight function on (a,b), H be a positive Green function associated to the linear differential operator L. Suppose that the function $x \mapsto u(x)\dfrac{H(x,t)}{\tilde{H}(x)}$ is integrable on (a,b) for each fixed $t \in (a,b)$, and that v is defined on (a,b) by (2.2). If $\Phi : (0,\infty) \to \mathrm{R}$ is convex and increasing function, then the inequality

$$\int_a^b u(x)\Phi\left(\left|\frac{1}{\tilde{H}(x)}\int_a^x H(x,t)h(t)dt\right|\right)dx$$
$$\le \int_a^b v(t)\Phi\left(|h(t)|\right)dt, \qquad (2.8)$$

holds for all measurable functions $h : (a,b) \to \mathrm{P}$, such that $Imh \subseteq I$ where $\tilde{H}$ is defined by (2.4).

Proof. Applying Theorem 1.3 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$ and $k(x,t) = H(x,t)$ we obtain inequality (2.8).

Remark 2.5. Choose the particular convex function $\Phi(x) = x^\nu$, $\nu \ge 1$ and weight function $u(x) = \tilde{H}(x)$ in Theorem 2.4 we get $v(t) = \int_t^b H(x,t)dx =: K_2(t)$ and

$$\int_a^b \tilde{H}(x)^{1-\nu}\left(\left|\int_a^x H(x,t)h(t)dt\right|\right)^\nu dx$$
$$\le \int_a^b K_2(t)\,|h(t)|^\nu\, dt. \qquad (2.9)$$

Inequality (2.9) gives

$$\tilde{H}(b)^{1-\nu}\int_a^b |y(x)|^\nu\, dx \le K_2(a)\int_a^b |h(t)|^\nu\, dt.$$

This implies that

$$\|y\|_\nu (a,b) \le \left(\frac{K_2(a)}{(\tilde{H}(b))^{1-\nu}}\right)^{\frac{1}{\nu}} \|h\|_\nu (a,b).$$

If we substitute $H(x,t)$ by $H(x,t)h_2(t)$ and h by $\dfrac{h_1}{h_2}$, where $h_i : (a,b) \to \mathrm{R}$, $(i = 1,2)$ are measurable functions in Theorem 2.4 we obtain the following result.

Theorem 2.6 Let u be a weight function on (a,b), H be a positive Green function associated to the linear differential operator L. Suppose that the function $x \mapsto u(x)\dfrac{H(x,t)}{y_2(x)}$ is integrable on (a,b) for each fixed $t \in (a,b)$, and that w is defined on (a,b) by (2.6). If $\Phi : (0,\infty) \to \mathrm{R}$ is convex and increasing fuctionn and $\dfrac{y_1(x)}{y_2(x)}, \dfrac{h_1(t)}{h_2(t)} \in I$, then the inequality

$$\int_a^b u(x)\Phi\left(\left|\frac{y_1(x)}{y_2(x)}\right|\right)dx \le \int_a^b w(t)\Phi\left(\left|\frac{h_1(t)}{h_2(t)}\right|\right)dt,$$

holds for all $y_i \in U(h_i, H), (i = 1,2)$ and for all measurable functions $h_i : (a,b) \to \mathrm{R}$, $(i = 1,2)$.

The upcoming theorem is the generalization of Theorem 2.1 for linear differential operator.

Theorem 2.7 Let u be a weight function on (a,b), H be a positive Green function associated to the linear differential operator L. Let $0 < p \le q < \infty$ and that the function $x \mapsto u(x)\left(\dfrac{H(x,t)}{\tilde{H}(x)}\right)^{\frac{q}{p}}$ is integrable on (a,b) for each fixed $t \in (a,b)$, and that v is defined on (a,b) by

$$v(t) := \left(\int_t^b u(x)\left(\frac{H(x,t)}{\tilde{H}(x)}\right)^{\frac{q}{p}} dx\right)^{\frac{p}{q}} < \infty.$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathrm{R}$, then the inequality

$$\left(\int_a^b u(x)\left[\Phi\left(\frac{1}{\tilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)\right]^{\frac{q}{p}} dx\right)^{\frac{1}{q}}$$

$$\leq \left(\int_a^b v(t)\Phi\left(h(t)\right)dt\right)^{\frac{1}{p}}, \qquad (2.10)$$

holds for all measurable functions $h:(a,b)\to \mathrm{R}$, such that $Imh \subseteq I$, where $\tilde{H}$ is defined by (2.4).

Proof. Applying Theorem 1.6 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$ and $k(x,t) = H(x,t)$, we obtain inequality (2.10).

Remark 2.8. Choose $\Phi(x) = x^{\nu}$, $\nu \geq 1$ and $u(x) = (\tilde{H}(x))^{\frac{q}{p}}$ in Theorem 2.7 we get $v(t) = \left(\int_t^b \left(H(x,t)\right)^{\frac{q}{p}} dx\right)^{\frac{p}{q}} =: K_3(t)$ and we obtain

$$\left(\int_a^b (\tilde{H}(x))^{(1-\nu)\frac{q}{p}}\left(\int_a^x H(x,t)h(t)dt\right)^{\frac{\nu q}{p}} dx\right)^{\frac{1}{q}}$$

$$\leq \left(\int_a^b K_3(t)h^{\nu}(t)dt\right)^{\frac{1}{p}}. \qquad (2.11)$$

Inequality (2.11) gives

$$\tilde{H}(b)^{1-\nu}\left(\int_a^b y^{\frac{\nu q}{p}}(x)dx\right)^{\frac{p}{q}} \leq K_3(a)\int_a^b h^{\nu}(t)dt.$$

Now we provide a new class of sufficient conditions on weight functions u and w, and on a Green function H, for a modular inequality involving linear differential operator in next theorem.

Theorem 2.9 Let $0 < p \leq q < \infty$. Let u be a weight function on (a,b), w be a positive function on (a,b), H be a positive measurable function on $(a,b)\times(a,b)$, and $\tilde{H}$ be defined on (a,b) by (2.4). Suppose that $\tilde{H}(x) > 0$ for all $x \in (a,b)$ and that the function $x \mapsto u(x)\left(\frac{H(x,t)}{\tilde{H}(x)}\right)^{\frac{q}{p}}$ is integrable on (a,b) for each fixed $t \in (a,b)$. Let Φ be a non-negative convex function on an interval $I \subseteq \mathrm{R}$. If

$$A = \sup_{t\in(a,b)} w^{\frac{-1}{p}}(t)\left(\int_t^b u(x)\left(\frac{H(x,t)}{\tilde{H}(x)}\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} < \infty,$$

then there exists a positive real constant C, such that the inequality

$$\left(\int_a^b u(x)\Phi^{\frac{q}{p}}\left(\frac{1}{\tilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)dx\right)^{\frac{1}{q}}$$

$$\leq C\left(\int_a^b w(t)\Phi(h(t))dt\right)^{\frac{1}{p}} \qquad (2.12)$$

holds for all measurable functions $h:(a,b)\to \mathrm{R}$ with values in I. Moreover, if C is the smallest constant for (2.12) to hold, then $C \leq A$.

Proof. Applying Theorem 1.7 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$ and $k(x,t) = H(x,t)$ we obtain inequality (2.12).

Now we provide two-parametric class of sufficient conditions for a weighted modular inequality involving linear differential operator to hold.

Theorem 2.10 Let $1 < p \leq q < \infty$. Let u be a weight function on (a,b), v be a measurable positive function on (a,b), H be a positive measurable function on $(a,b)\times(a,b)$, and $\tilde{H}$ be defined on (a,b) by (2.4). Let $\tilde{H}(x) > 0$ for all $x \in (a,b)$ and let the function $x \mapsto u(x)\left(\frac{H(x,t)}{\tilde{H}(x)}\right)^{q}$ be integrable on (a,b) for each fixed $t \in (a,b)$. Suppose that $\Phi: I \to [0,\infty)$ is a bijective convex function on an interval $I \subseteq \mathrm{R}$. If there exist a real parameter $s \in (1,p)$ and a positive measurable function $V:(a,b)\to \mathrm{R}$ such that

$$A(s,V) = F(V,v)\sup_{t\in(a,b)} V^{\frac{s-1}{p}}(t)$$

$$\times\left[\int_t^b u(x)\left(\frac{H(x,t)}{\tilde{H}(x)}\right)^{q} dx\right]^{\frac{1}{q}} < \infty,$$

where

$$F(V,v) = \left(\int_a^b V^{\frac{-p'(s-1)}{p}}(t)v^{1-p'}(t)dt\right)^{\frac{1}{p'}},$$

then there is a positive real constant C such that the inequality

$$\left(\int_a^b u(x)\Phi^q\left(\frac{1}{\tilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)dx\right)^{\frac{1}{q}}$$

$$\leq C\left(\int_a^b v(t)\Phi^p(h(t))dt\right)^{\frac{1}{p}} \qquad (2.13)$$

holds for all measurable functions $h:(a,b)\to \mathrm{R}$ with values in I. Moreover, if C is the best possible constant in (2.13), then

$$C \leq \inf_{\substack{1<s<p \\ V>0}} A(s,V). \qquad (2.14)$$

Proof. Applying Theorem 1.8 with $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx$, $d\mu_2(t)=dt$ and $k(x,t)=H(x,t)$ we obtain inequality (2.14).

By modifying Theorem 2.10, we obtain the following result.

Theorem 2.11 Let $1<p\leq q<\infty$, $1<s<p$, and $0<b\leq\infty$. Let u be a weight function on $(0,b)$, w be a positive measurable function on $(0,b)$, and H be a positive measurable function on $(0,b)\times(0,b)$. Let I be an interval in R and $\Phi: I\to[0,\infty)$ be a bijective convex function. If

$$V(t)=\int_0^t w^{1-p'}(x)x^{p'-1}dx<\infty$$

holds almost everywhere in $(0,b)$ and

$$A(s)=\sup_{0<t<b}\left(\int_t^b u(x)\left(\frac{H(x,t)}{\tilde{H}(x)}\right)^q V^{\frac{q(p-s)}{p}}(x)\frac{dx}{x}\right)^{\frac{1}{q}}$$

$$\times V^{\frac{s-1}{p}}(t)<\infty,$$

there exists a positive real constant C such that

$$\left(\int_0^b u(x)\Phi^q\left(\frac{1}{\tilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)\frac{dx}{x}\right)^{\frac{1}{q}}$$

$$\leq C\left(\int_0^b w(x)\Phi^p(h(x))\frac{dx}{x}\right)^{\frac{1}{p}} \qquad (2.15)$$

holds for all measurable functions $h:(0,b)\to\mathrm{R}$ with values in I. Moreover, if C is the best possible constant in (2.15), then

$$C\leq \inf_{1<s<p}\left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}}A(s).$$

Proof. Applying Theorem 1.9 with $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx$, $d\mu_2(t)=dt$ and $k(x,t)=H(x,t)$ we obtain inequality (2.15).

Now we give refined Hardy-type inequalities involving linear differential operator for an arbitrary non-negative convex function in the following theorem.

Theorem 2.12 Let $\tau\in\mathrm{R}_+$, u be a weight function on Ω_1, H a non-negative measurable function on $(a,b)\times(a,b)$, and $\tilde{H}$ be defined on (a,b) by (2.4). Suppose that $\tilde{H}(x)>0$ for all $x\in(a,b)$, that the function $x\mapsto u(x)\left(\frac{H(x,t)}{\tilde{H}(x)}\right)^{\tau}$ is integrable on (a,b) for each fixed $t\in(a,b)$, and that v is defined on (a,b) by

$$v(t)=\left(\int_t^b u(x)\left(\frac{H(x,t)}{\tilde{H}(x)}\right)^{\tau}dx\right)^{\frac{1}{\tau}}.$$

If Φ is a non-negative convex function on an interval $I\subseteq\mathrm{R}$ and $\Phi: I\to\mathrm{R}$ is any function, such that $\varphi(x)\in\partial\Phi(x)$ for all $x\in \mathrm{Int}I$, then the inequality

$$\left(\int_a^b v(t)\Phi(f(t))dt\right)^{\tau}-\int_a^b u(x)\Phi^{\tau}\left(\frac{1}{\tilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)dx$$

$$\geq\tau\int_a^b \frac{u(x)}{\tilde{H}(x)}\Phi^{\tau-1}\left(\frac{1}{\tilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)$$

$$\times\int_a^b H(x,t)r(x,t)dtdx \qquad (2.16)$$

holds for all $\tau\geq1$ and all measurable functions $h:(a,b)\to\mathrm{R}$ with values in I, and the function $r:(a,b)\times(a,b)\to R$ is defined by

$$r(x,t)=\left\|\Phi(h(t))-\Phi\left(\frac{1}{\tilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)\right|$$

$$-\left|\varphi\left(\frac{1}{\tilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)\right|\cdot\left|h(t)-\frac{1}{\tilde{H}(x)}\int_a^x H(x,t)h(t)dt\right\|. \qquad (2.17)$$

If $\tau\in(0,1]$ and the function $\Phi: I\to\mathrm{R}$ is positive and concave, then the order of terms on the left-hand side of (2.16) is reversed, that is, the inequality

$$\int_a^b u(x)\Phi^{\tau}\left(\frac{1}{\tilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)dx-\left(\int_a^b v(t)\Phi(h(t))dt\right)^{\tau}$$

$$\geq\tau\int_a^b \frac{u(x)}{\tilde{H}(x)}\Phi^{\tau-1}\left(\frac{1}{\tilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)$$

$$\times\int_a^b H(x,t)r(x,t)dtdx \qquad (2.18)$$

holds.

Let the function $r_1:(a,b)\times(a,b)\to \mathrm{R}$ is defined by

$$r_1(x,t)=\Phi(h(t))-\Phi\left(\frac{1}{\tilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)$$
$$-\left|\varphi\left(\frac{1}{\tilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)\right|\cdot\left(h(t)-\frac{1}{\tilde{H}(x)}\int_a^x H(x,t)h(t)dt\right). \quad (2.19)$$

If Φ is non-negative monotone convex on the interval $I\subseteq \mathrm{R}$, and $\varphi: I\to \mathrm{R}$ is any function, such that $\varphi(x)\in\partial\Phi(x)$ for all $x\in Int I$, then the inequality

$$\left(\int_a^b v(t)\Phi\left(h(t)\right)dy\right)^{\tau}$$
$$-\int_a^b u(x)\Phi^{\tau}\left(\frac{1}{\tilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)dx$$
$$\geq \tau\,|\int_a^b \frac{u(x)}{\tilde{H}(x)}\Phi^{\tau-1}\left(\frac{1}{\tilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)$$
$$\times\int_a^b sgn\left(h(t)-\frac{1}{\tilde{H}(x)}\int_a^x H(x,t)h(t)dt\right)$$
$$\times H(x,t)r_1(x,t)dtdx\,| \quad (2.20)$$

holds for all measurable functions $h:(a,b)\to \mathrm{R}$, such that $h(t)\in I$, for all fixed $t\in(a,b)$.

If Φ is non-negative monotone concave, then the order of terms on the left-hand side of (2.20) is reversed.

Proof. Applying Theorem 1.10 with $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx$, $d\mu_2(t)=dt$ and $k(x,t)=H(x,t)$ we obtain inequalities (2.16) and (2.20).

We start with the standard one-dimensional setting, that is, by considering intervals in R and the Lebesgue measure, and obtain generalized refined Hardy and Pólya–Knopp–type inequalities for linear differential operator.

Theorem 2.13 Let $0<b\leq\infty$ and $k:(0,b)\times(0,b)\to \mathrm{R}$, $u:(0,b)\to \mathrm{R}$ be non-negative measurable functions and

$$w(t)=t\left(\int_t^b u(x)\left(\frac{H(x,t)}{\tilde{H}(x)}\right)^{\frac{q}{p}}\frac{dx}{x}\right)^{\frac{p}{q}}<\infty, t\in(0,b).$$

If $0<p\leq q<\infty$ or $-\infty<q\leq p<0$, Φ is a non-negative convex function on an interval $I\subseteq \mathrm{R}$, and $\varphi: I\to \mathrm{R}$ is such that $\varphi(x)\in\partial\Phi(x)$ for all $x\in \mathrm{Int}I$, then the inequality

$$\left(\int_0^b w(t)\Phi(h(t))\frac{dt}{t}\right)^{\frac{q}{p}}-\int_0^b u(x)\Phi^{\frac{q}{p}}\left(\frac{1}{\tilde{H}(x)}\int_0^x H(x,t)h(t)dt\right)\frac{dx}{x}$$
$$\geq\frac{q}{p}\int_0^b \frac{u(x)}{\tilde{H}(x)}\Phi^{\frac{q}{p}-1}\left(\frac{1}{\tilde{H}(x)}\int_0^x H(x,t)h(t)dt\right)$$
$$\times\int_0^x H(x,t)r(x,t)dt\frac{dx}{x} \quad (2.21)$$

holds for all measurable functions $h:(0,b)\to \mathrm{R}$ with values in I and r is defined by (2.17). If Φ is non-negative monotone convex on the interval $I\subseteq \mathrm{R}$ and $\varphi: I\to \mathrm{R}$ is that $\varphi(x)\in\partial\Phi(x)$ for all $x\in IntI$, then the following inequality

$$\left(\int_0^b w(t)\Phi\left(h(t)\right)\frac{dt}{t}\right)^{\frac{q}{p}}$$
$$-\int_0^b u(x)\Phi^{\frac{q}{p}}\left(\frac{1}{\tilde{H}(x)}\int_0^x H(x,t)h(t)dt\right)\frac{dx}{x}$$
$$\geq\frac{q}{p}\,|\int_0^b \frac{u(x)}{\tilde{H}(x)}\Phi^{\frac{q}{p}-1}\left(\frac{1}{\tilde{H}(x)}\int_0^x H(x,t)h(t)dt\right)$$
$$\times\int_0^x sgn\left(h(t)-\frac{1}{\tilde{H}(x)}\int_0^x H(x,t)h(t)dt\right)$$
$$\times H(x,t)r_1(x,t)dt\frac{dx}{x}\,| \quad (2.22)$$

holds for all measurable functions $h:(0,b)\to \mathrm{R}$, such that $h(t)\in I$, for all fixed $t\in(0,b)$ and r_1 is defined by (2.19).

If $0<q\leq p<\infty$ or $-\infty<p\leq q<0$, and Φ is a non-negative (monotone) concave function, then (2.21) and (2.22) hold with the reversed order of integrals on its left-hand side.

Proof. Applying Theorem 1.11 with $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx$, $d\mu_2(t)=dt$ and $k(x,t)=H(x,t)$ we obtain inequalities (2.21) and (2.22).

3.0 HARDY-TYPE INEQUALITIES FOR WIDDER;S DERIVATIVE

First it is necessary to give some important details about Widder's derivatives (see[8]).

Let $f,u_0,u_1,\ldots,u_n\in C^{n+1}[a,b], n\geq 0,$ and the Wronskians

$$W_i(x) := W[u_0(x), u_1(x), \ldots, u_i(x)]$$

$$= \begin{vmatrix} u_0(x) & \cdot & \cdot & \cdot & u_i(x) \\ u_0'(x) & \cdot & \cdot & \cdot & u_i'(x) \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ u_0^{(i)}(x) & \cdot & \cdot & \cdot & u_i^{(i)}(x) \end{vmatrix},$$

$i = 0,1,\ldots,n.$ Here $W_0(x) = u_0(x).$ Assume $W_i(x) > 0$ over $[a,b], i = 0,1,\ldots,n.$ For $i \geq 0,$ the differential operator of order i (Widder derivative):

$$L_i f(x) := \frac{W[u_0(x), u_1(x), \ldots, u_{i-1}(x), f(x)]}{W_{i-1}(x)},$$

$i = 1,\ldots,n+1; L_0 f(x) = f(x)$

for all $x \in [a,b].$ Consider also

$$g_i(x,t) := \frac{1}{W_i(t)} \begin{vmatrix} u_0(t) & \cdot & \cdot & \cdot & u_i(t) \\ u_0'(t) & \cdot & \cdot & \cdot & u_i'(t) \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ u_0(x) & \cdot & \cdot & \cdot & u_i(x) \end{vmatrix},$$

$i = 1,2,...,n;\ g_0(x,t) := \dfrac{u_0(x)}{u_0(t)}$

for all $x,t \in [a,b].$

Example 3.1 [8]. Sets of the form $\{u_0, u_1, u_2, \ldots, u_n\}$ are $\{1, x, x^2, \ldots, x^n\},$

$\{1, \sin x, -\cos x, -\sin 2x, \cos 2x, \ldots, (-1)^{n-1} \sin nx, (-1)^{n-1} \cos nx\},$ *etc*

We also mention the generalized Widder-Taylor's formula, see [8](see also [1]).

Theorem 3.2 Let the functions $f, u_0, u_1, \ldots, u_n \in C^{n+1}[a,b],$ and the Wronkians $W_0(x), W_1(x), \ldots, W_n(x) > 0$ on $[a,b], x \in [a,b].$ Then for $t \in [a,b]$ we have

$$f(x) = f(t)\frac{u_0(x)}{u_0(t)} + L_1 f(t) g_1(x,t) + \ldots$$
$$+ L_n f(t) g_n(x,t) + R_n(x),$$

where

$$R_n(x) := \int_t^x g_n(x,s) L_{n+1} f(s) ds.$$

For example (see [8]) one could take $u_0(x) = c > 0.$ If $u_i(x) = x^i, i = 0,1,\ldots,n,$ defined on [a,b], then

$$L_i f(t) = f^{(i)}(t) \text{ and } g_i(x,t) = \frac{(x-t)^i}{i!}, \quad t \in [a,b].$$

We need the following corollary.

Corollary 3.3 By additionally assuming for fixed a that $L_i f(a) = 0, i = 0,1,...,n,$ we get that

$$f(x) := \int_a^x g_n(x,t) L_{n+1} f(t) dt \quad \text{for all } x \in [a,b].$$

The proofs of all results in this section can be completed by taking $\Omega_1 = \Omega_2 = (a,b),\ d\mu_1(x) = dx,\ d\mu_2(t) = dt$ and $k(x,t) = g_n(x,t)$ in all theorems given in Section 1 but we omit the details.

Now we prove Hardy-type inequalities for Widder's derivative.

Theorem 3.4 Let the assumptions of the Corollary 3.3 be satisfied. Let u be a weight function on $(a,b),$ g_n be a positive measurable function on $(a,b) \times (a,b).$ Suppose that the function $x \mapsto u(x)\dfrac{g_n(x,t)}{\tilde{g}_n(x)}$ is integrable on (a,b) for each fixed $t \in (a,b),$ and that v be defined on (a,b) by

$$v(t) := \int_t^b \frac{u(x) g_n(x,t)}{\tilde{g}_n(x)} dx < \infty. \tag{3.1}$$

If Φ is convex on the interval $I \subseteq \mathrm{R},$ then the inequality

$$\int_a^b u(x) \Phi\left(\frac{1}{\tilde{g}_n(x)} \int_a^x g_n(x,t) L_{n+1} f(t) dt \right) dx$$
$$\leq \int_a^b v(t) \Phi\left(L_{n+1} f(t) \right) dt,$$

holds for all measurable functions $L_{n+1} f : (a,b) \to \mathrm{R},$ such that $Im L_{n+1} f \subseteq I,$ where $\tilde{g}_n$ is defined as

$$\tilde{g}_n(x) := \int_a^x g_n(x,t) dx < \infty. \tag{3.2}$$

Remark 3.5 Choose the particular convex function $\Phi(x) = x^\nu,\ \nu \geq 1$ and weight function $u(x) = \tilde{g}_n(x)$ in Theorem 3.4 we get $v(t) = \int_t^b g_n(x,t) dx =: K_4(t)$ and we obtain

$$\int_a^b \tilde{g}_n(x)^{1-\nu}\left(\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)^{\nu}dx$$
$$\leq \int_a^b K_4(t)L_{n+1}f^{\nu}(t)dt. \qquad (3.3)$$

Inequality (3.3) gives

$$\tilde{g}_n(b)^{1-\nu}\int_a^b f^{\nu}(x)dx \leq K_4(a)\int_a^b L_{n+1}f^{\nu}(t)dt.$$

We have that

$$\|f\|_{\nu}(a,b) \leq \left(\frac{K_4(a)}{(\tilde{g}_n(b))^{1-\nu}}\right)^{\frac{1}{\nu}}\|L_{n+1}f\|_{\nu}(a,b).$$

Now we give the Hardy-type inequality involving Widder's derivative in quotients. For this if we substitute $g_n(x,t)$ by $g_n(x,t)L_{n+1}f_2(t)$ and f by $\frac{L_{n+1}f_1}{L_{n+1}f_2}$, where $L_{n+1}f_i:(a,b)\to R,\ (i=1,2)$ are measurable functions in Theorem 3.4 we obtain the following result.

Theorem 3.6 Let the assumptions of the Corollary 3.3 be satisfied. Let u be a weight function on (a,b), g_n be a positive measurable function on $(a,b)\times(a,b)$. Assume that the function $x\mapsto u(x)\frac{g_n(x,t)}{L_{n+1}f_2(x)}$ is integrable on (a,b) for each fixed $t\in(a,b)$. Define v on (a,b) by

$$v(t):=L_{n+1}f_2(t)\int_t^b u(x)\frac{g_n(x,t)}{L_{n+1}f_2(x)}dx<\infty. \qquad (3.4)$$

If $\Phi:I\to \mathrm{R}$ is a convex function and $\frac{f_1(x)}{f_2(x)},\frac{L_{n+1}f_1(t)}{L_{n+1}f_2(t)}\in I$, then the inequality

$$\int_a^b u(x)\Phi\left(\frac{f_1(x)}{f_2(x)}\right)dx \leq \int_a^b v(t)\Phi\left(\frac{L_{n+1}f_1(t)}{L_{n+1}f_2(t)}\right)dt,$$

holds for all $f_i\in U(L_{n+1}f_i,g_n),(i=1,2)$ and for all measurable functions $L_{n+1}f_i:(a,b)\to\mathrm{R},.\quad (i=1,2)$.

Now we prove Hardy-type inequalities for Widder's derivative involving convex and increasing function.

Theorem 3.7 Let the assumptions of the Corollary 3.3 be satisfied. Let u be a weight function on (a,b), g_n be a positive measurable function on $(a,b)\times(a,b)$. Suppose that the function $x\mapsto u(x)\frac{g(x,t)}{\tilde{g}_n(x)}$ is integrable on (a,b) for each fixed $t\in(a,b)$, and that v is defined on (a,b) by (3.1). If Φ is convex and increasing on the interval $I\subseteq\mathrm{R}$, then the inequality

$$\int_a^b u(x)\Phi\left(\left|\frac{1}{\tilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right|\right)dx$$
$$\leq \int_a^b v(t)\Phi\left(|L_{n+1}f(t)|\right)dt,$$

holds for all measurable functions $L_{n+1}f:(a,b)\to\mathrm{R}$, such that $ImL_{n+1}f\subseteq I$, where $\tilde{g}_n$ is defined by (3.2).

Remark 3.8 Choose the particular convex function $\Phi(x)=x^{\nu},\ \nu\geq 1$ and weight function $u(x)=\tilde{g}_n(x)$ in Theorem 3.7 we get $v(t)=\int_t^b g_n(x,t)dx=:K_5(t)$ and we obtain

$$\int_a^b \tilde{g}_n(x)^{1-\nu}\left(\left|\int_a^x g_n(x,t)L_{n+1}f(t)dt\right|\right)^{\nu}dx$$
$$\leq \int_a^b K_5(t)\,|L_{n+1}f^{\nu}(t)|\,dt. \qquad (3.5)$$

Inequality (3.5) gives

$$\tilde{g}_n(b)^{1-\nu}\int_a^b |f(x)|^{\nu}dx \leq K_5(a)\int_a^b |L_{n+1}f(t)^{\nu}|\,dt.$$

We have that

$$\|f\|_{\nu}(a,b)\leq\left(\frac{K_5(a)}{(g_n(b))^{1-\nu}}\right)^{\frac{1}{\nu}}\|L_{n+1}f\|_{\nu}(a,b).$$

Now we give the Hardy-type inequality involving Widder's derivative in quotients for convex and increasing function. For this if we substitute $g_n(x,t)$ by $g_n(x,t)L_{n+1}f_2(t)$ and f by $\frac{L_{n+1}f_1}{L_{n+1}f_2}$, where $L_{n+1}f_i:(a,b)\to\mathrm{R},(i=1,2)$ are measurable functions in Theorem 3.7 we obtain the following result.

Theorem 3.9 Let the assumptions of the Corollary 3.3 be satisfied. Let u be a weight function on (a,b), g_n be a positive measurable function on $(a,b)\times(a,b)$. Assume that the function $x\mapsto u(x)\frac{g_n(x,t)}{L_{n+1}f_2(x)}$ is integrable on (a,b) for each fixed $t\in(a,b)$ and define v on (a,b) by (3.4). If $\Phi:I\to\mathrm{R}$ is a convex and increasing function and $\frac{f_1(x)}{f_2(x)},\frac{L_{n+1}f_1(t)}{L_{n+1}f_2(t)}\in I$, then the inequality

$$\int_a^b u(x)\Phi\left(\left|\frac{f_1(x)}{f_2(x)}\right|\right)dx\leq\int_a^b v(t)\Phi\left(\left|\frac{L_{n+1}f_1(t)}{L_{n+1}f_2(t)}\right|\right)dt,$$

holds for all $f_i \in U(L_{n+1} f_i, g_n), (i=1,2)$ and for all measurable functions $L_{n+1} f_i : (a,b) \to \mathbb{R},.\quad (i=1,2)$.

The upcoming theorem is the generalization of Theorem 3.4.

Theorem 3.10 Let the assumptions of the Corollary 3.3 be satisfied. Let u be a weight function on (a,b) and g_n be a positive measurable function on $(a,b)\times(a,b)$. Let $0<p\le q<\infty$ and that the function $x \mapsto u(x)\left(\frac{g_n(x,t)}{\tilde{g}_n(x)}\right)^{\frac{q}{p}}$ is integrable on (a,b) for each fixed $t\in(a,b)$, and that v is defined on (a,b) by

$$v(t) := \left(\int_t^b u(x)\left(\frac{g_n(x,t)}{\tilde{g}_n(x)}\right)^{\frac{q}{p}} dx\right)^{\frac{p}{q}} < \infty.$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_a^b u(x)\left[\Phi\left(\frac{1}{\tilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)\right]^{\frac{q}{p}} dx\right)^{\frac{1}{q}}$$

$$\le \left(\int_a^b v(t)\Phi\left(L_{n+1}f(t)\right)dt\right)^{\frac{1}{p}},$$

holds for all measurable functions $L_{n+1}f : (a,b) \to \mathrm{P}$, such that $ImL_{n+1}f \subseteq I$, where $\tilde{g}_n$ is defined by (3.2).

Remark 3.11 Choose $\Phi(x) = x^{\nu}$, $\nu \ge 1$ and $u(x) = (\tilde{g}_n(x))^{\frac{q}{p}}$ in Theorem 3.10 we get $v(t) = \left(\int_t^b (g_n(x,t))^{\frac{q}{p}} dx\right)^{\frac{p}{q}} =: K_6(t)$ and we obtain

$$\left(\int_a^b (\tilde{g}_n(x))^{(1-\nu)\frac{q}{p}}\left(\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)^{\frac{\nu q}{p}} dx\right)^{\frac{1}{q}}$$

$$\le \left(\int_a^b K_6(t)L_{n+1}f^{\nu}(t)dt\right)^{\frac{1}{p}}. \tag{3.6}$$

Inequality (3.6) gives

$$\tilde{g}_n(b)^{1-\nu}\left(\int_a^b f^{\frac{\nu q}{p}}(x)dx\right)^{\frac{p}{q}} \le K_6(a)\int_a^b L_{n+1}f^{\nu}(t)dt.$$

Here a new class of sufficient conditions on weight functions u and w, and on g_n, for a modular inequality involving is given.

Theorem 3.12 Let $0<p\le q<\infty$. Let u be a weight function on (a,b), w be a positive function on (a,b), g_n be a positive measurable function on $(a,b)\times(a,b)$, and $\tilde{g}_n$ be defined on (a,b) by (3.2). Suppose that $\tilde{g}_n(x)>0$ for all $x\in(a,b)$ and that the function $x \mapsto u(x)\left(\frac{g_n(x,t)}{\tilde{g}_n(x)}\right)^{\frac{q}{p}}$ is integrable on (a,b) for each fixed $t\in(a,b)$. Let Φ be a non-negative convex function on an interval $I\subseteq\mathbb{R}$. If

$$A = \sup_{t\in(a,b)} w^{-\frac{1}{p}}(t)\left(\int_y^b u(x)\left(\frac{g_n(x,t)}{\tilde{g}_n(x)}\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} < \infty,$$

then there exists a positive real constant C, such that the inequality

$$\left(\int_a^b u(x)\Phi^{\frac{q}{p}}\left(\frac{1}{\tilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)dx\right)^{\frac{1}{q}}$$

$$\le C\left(\int_a^b w(t)\Phi(L_{n+1}f(t))dt\right)^{\frac{1}{p}} \tag{3.7}$$

holds for all measurable functions $L_{n+1}f : (a,b)\to\mathbb{R}$ with values in I. Moreover, if C is the smallest constant for (3.7) to hold, then $C \le A$.

We continue by providing a new two-parametric class of sufficient conditions for a weighted modular inequality involving Widder's derivative.

Theorem 3.13 Let $1<p\le q<\infty$. Let u be a weight function on (a,b), v be a measurable positive function on (a,b), g_n be a positive measurable function on $(a,b)\times(a,b)$, and $\tilde{g}_n$ be defined on (a,b) by (3.2). Let $\tilde{g}_n(x)>0$ for all $x\in(a,b)$ and let the function $x \mapsto u(x)\left(\frac{g_n(x,t)}{\tilde{g}_n(x)}\right)^{q}$ be integrable on (a,b) for each fixed $t\in(a,b)$. Suppose that $\Phi: I\to[0,\infty)$ is a bijective convex function on an interval $I\subseteq\mathbb{R}$. If there exist a real parameter $s\in(1,p)$ and a positive measurable function $V:(a,b)\to\mathbb{R}$ such that

$$A(s,V)=F(V,v)\sup_{t\in(a,b)}V^{\frac{s-1}{p}}(t)$$

$$\times\left[\int_a^b u(x)\left(\frac{g_n(x,t)}{\tilde{g}_n(x)}\right)^q dx\right]^{\frac{1}{q}}<\infty,$$

where

$$F(V,v)=\left(\int_a^b V^{\frac{-p'(s-1)}{p}}(t)v^{1-p'}(t)dt\right)^{\frac{1}{p'}},$$

then there is a positive real constant C such that the inequality

$$\left(\int_a^b u(x)\Phi^q\left(\frac{1}{\tilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)dx\right)^{\frac{1}{q}}$$

$$\leq C\left(\int_a^b v(t)\Phi^p(L_{n+1}f(t))dt\right)^{\frac{1}{p}} \qquad (3.8)$$

holds for all measurable functions $L_{n+1}f:(a,b)\to\mathrm{R}$ with values in I. Moreover, if C is the best possible constant in (3.8), then

$$C\leq\inf_{\substack{1<s<p\\ V>0}} A(s,V).$$

By modifying Theorem 3.13, we obtain the following result.

Theorem 3.14 Let $1<p\leq q<\infty$, $1<s<p$, and $0<b\leq\infty$. Let u be a weight function on $(0,b)$, w be a positive measurable function on $(0,b)$, and g_n be a positive measurable function on $(0,b)\times(0,b)$. Let I be an interval in R and $\Phi:I\to[0,\infty)$ be a bijective convex function. If

$$V(t)=\int_0^t w^{1-p'}(x)x^{p'-1}dx<\infty$$

holds almost everywhere in $(0,b)$ and

$$A(s)=\sup_{0<t<b}\left(\int_t^b u(x)\left(\frac{g_n(x,t)}{\tilde{g}_n(x)}\right)^q V^{\frac{q(p-s)}{p}}(x)\frac{dx}{x}\right)^{\frac{1}{q}}$$

$$\times V^{\frac{s-1}{p}}(t)<\infty,$$

then there exists a positive real constant C such that

$$\left(\int_0^b u(x)\Phi^q\left(\frac{1}{\tilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)\frac{dx}{x}\right)^{\frac{1}{q}}$$

$$\leq C\left(\int_0^b w(x)\Phi^p(f(x))\frac{dx}{x}\right)^{\frac{1}{p}} \qquad (3.9)$$

holds for all measurable functions $L_{n+1}f:(0,b)\to\mathrm{R}$ with values in I. Moreover, if C is the best possible constant in (3.9), then

$$C\leq\inf_{1<s<p}\left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}}A(s).$$

The rest of this section is dedicated to new refined inequalities. A new general refined weighted Hardy-type inequality with g_n related to an arbitrary non-negative convex function is given in the following theorem.

Theorem 3.15 Let $\tau\in R_+$, u be a weight function on (a,b), g_n a positive measurable function on $(a,b)\times(a,b)$, and $\tilde{g}_n$ be defined on (a,b) by (3.2). Suppose that $\tilde{g}_n(x)>0$ for all $x\in(a,b)$, that the function $x\mapsto u(x)\left(\frac{g_n(x,t)}{\tilde{g}_n(x)}\right)^{\tau}$ is integrable on (a,b) for each fixed $t\in(a,b)$, and that v is defined on (a,b) by

$$v(t)=\left(\int_y^b u(x)\left(\frac{g_n(x,t)}{\tilde{g}_n(x)}\right)^{\tau}dx\right)^{\frac{1}{\tau}}.$$

If Φ is a non-negative convex function on an interval $I\subseteq\mathrm{R}$ and $\varphi:I\to\mathrm{R}$ is any function, such that $\varphi(x)\in\partial\Phi(x)$ for all $x\in\mathrm{Int}I$, then the inequality

$$\left(\int_a^b v(t)\Phi(L_{n+1}f(t))dt\right)^{\tau}$$

$$-\int_a^b u(x)\Phi^{\tau}\left(\frac{1}{\tilde{g}(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)dx$$

$$\geq\tau\int_a^b\frac{u(x)}{\tilde{g}_n(x)}\Phi^{\tau-1}\left(\frac{1}{\tilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)$$

$$\times\int_a^b H(x,t)r(x,t)dtdx$$

holds for all $\tau\geq1$ and all measurable functions $L_{n+1}f:(a,b)\to\mathrm{R}$ with values in I, and the function $r:(a,b)\times(a,b)\to\mathrm{R}$ is defined by

$$r(x,t)=\left\|\Phi(L_{n+1}f(t))-\Phi\left(\frac{1}{\tilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)\right|$$

$$-\left|\varphi\left(\frac{1}{\tilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)\right|\cdot\left|L_{n+1}f(t)-\frac{1}{\tilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right\|.(3.11)$$

If $\tau \in (0,1]$ and the function $\Phi : I \to \mathrm{P}$ is positive and concave, then the order of terms on the left-hand side of (3.10) is reversed, that is, the inequality

$$\int_a^b u(x)\Phi^{\tau}\left(\frac{1}{\tilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)dx$$
$$-\left(\int_a^b v(t)\Phi(L_{n+1}f(t))dt\right)^{\tau}$$
$$\geq \tau\int_a^b \frac{u(x)}{\tilde{g}_n(x)}\Phi^{\tau-1}\left(\frac{1}{\tilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)$$
$$\times\int_a^b g_n(x,t)r(x,t)dtdx$$

holds.

Let the function $r_1 : (a,b)\times(a,b) \to \mathrm{P}$ is defined by

$$r_1(x,t) = \Phi(L_{n+1}f(t))$$
$$-\Phi\left(\frac{1}{\tilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)$$
$$-\left|\varphi\left(\frac{1}{\tilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)\right|$$
$$\times\left(L_{n+1}f(t)-\frac{1}{\tilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right). \qquad (3.12)$$

If Φ is non-negative monotone convex on the interval $I \subseteq \mathrm{R},$ and $\varphi : I \to \mathrm{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in IntI,$ then the inequality

$$\left(\int_a^b v(t)\Phi\left(L_{n+1}f(t)\right)dt\right)^{\tau}$$
$$-\int_a^b u(x)\Phi^{\tau}\left(\frac{1}{\tilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)dx$$
$$\geq \tau\left|\int_a^b \frac{u(x)}{\tilde{g}_n(x)}\Phi^{\tau-1}\left(\frac{1}{\tilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)\right.$$
$$\left.\times\int_a^b sgn\left(L_{n+1}f(t)-\frac{1}{\tilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)g_n(x,t)r_1(x,t)dtdx\right| \qquad (3.13)$$

holds for all measurable functions $L_{n+1}f : (a,b) \to \mathrm{R},$ such that $L_{n+1}f(t) \in I,$ for all fixed $t \in (a,b).$

If Φ is non-negative monotone concave, then the order of terms on the left-hand side of (3.13) is reversed.

We give the standard one-dimensional setting, that is, by considering intervals in R and the Lebesgue measure, and obtain generalized refined Hardy and Pólya–Knopp–type inequalities involving Widder's derivative.

Theorem 3.16 Let $0 < b \leq \infty$ and $g_n : (0,b)\times(0,b) \to \mathrm{R}$, $u : (0,b) \to \mathrm{R}$ be non-negative measurable functions and

$$w(t) = t\left(\int_t^b u(x)\left(\frac{g_n(x,t)}{\tilde{g}_n(x)}\right)^{\frac{q}{p}}\frac{dx}{x}\right)^{\frac{p}{q}} < \infty, t \in (0,b).$$ If $0 < p \leq q < \infty$ or $-\infty < q \leq p < 0$, Φ is a non-negative convex function on an interval $I \subseteq \mathrm{R}$, and $\varphi : I \to \mathrm{R}$ is such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \mathrm{Int}I$, then the inequality

$$\left(\int_0^b w(t)\Phi(L_{n+1}f(t))\frac{dt}{t}\right)^{\frac{q}{p}}$$
$$-\int_0^b u(x)\Phi^{\frac{q}{p}}\left(\frac{1}{\tilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)\frac{dx}{x}$$
$$\geq \frac{q}{p}\int_0^b \frac{u(x)}{\tilde{g}_n(x)}\Phi^{\frac{q}{p}-1}\left(\frac{1}{\tilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)$$
$$\times\int_0^x g_n(x,t)r(x,t)dt\frac{dx}{x} \qquad (3.14)$$

holds for all measurable functions $L_{n+1}f : (0,b) \to \mathrm{R}$ with values in I and r is defined by (3.11). If Φ is non-negative monotone convex on the interval $I \subseteq \mathrm{R}$ and $\varphi : I \to \mathrm{R}$ is that $\varphi(x) \in \partial\Phi(x)$ for all $x \in IntI,$ then the following inequality

$$\left(\int_0^b w(t)\Phi\left(L_{n+1}f(t)\right)\frac{dt}{t}\right)^{\frac{q}{p}}$$
$$-\int_0^b u(x)\Phi^{\frac{q}{p}}\left(\frac{1}{\tilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)\frac{dx}{x}$$
$$\geq \frac{q}{p}\left|\int_0^b \frac{u(x)}{\tilde{g}_n(x)}\Phi^{\frac{q}{p}-1}\left(\frac{1}{\tilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)\right. \qquad (3.13)$$
$$\left.\times\int_0^x sgn\left(L_{n+1}f(t)-\frac{1}{\tilde{g}_n(x)}\int_a^x g_n(x,t)L_{n+1}f(t)dt\right)g_n(x,t)r_1(x,t)dt\frac{dx}{x}\right| (3.15)$$

holds for all measurable functions $L_{n+1}f : (0,b) \to \mathrm{R},$ such that $L_{n+1}f(t) \in I,$ for all fixed $t \in (0,b)$ where r_1 is defined (3.12).

If $0 < q \le p < \infty$ or $-\infty < p \le q < 0$, and Φ is a non-negative (monotone) concave function, then (3.14) and (3.15) hold with the reversed order of integrals on its left-hand side.

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